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# $B$-Splines, Hypergeometric Functions, and Dirichlet Averages*

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Properties of Dirichlet averages are used to derive some well-known and some new properties of univariate and multivariate simplex splines. A univariate $B$-spline is the jump discontinuity of a hypergeometric $R$-function, and the Fourier transform of a $B$-spline is a confluent hypergeometric $S$-function. A univariate or multivariate $B$-spline is a Dirichlet average of a Dirac delta function. Its dependence on the knots is governed by a system of Euler–Poisson partial differential equations.


## 1. INTRODUCTION

A univariate $B$-spline $M(x) = M(x \mid x_0, ..., x_k)$ with knots $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_k$, where $x_0 < x_k$, is a real piecewise polynomial in $x$ of degree $k-1$. It is strictly positive if $x_0 < x < x_k$ and vanishes if $x < x_0$ or $x > x_k$. At a point where $m$ knots coincide, the derivative of order $k-m$ is discontinuous but derivatives of lower order are continuous. The knots determine the spline up to a normalizing constant, and the normalization of Curry and Schoenberg [5] is convenient in the present context:

$$\int_{-\infty}^{\infty} M(x)\, dx = 1. \tag{1.1}$$

Given the knots of a $B$-spline, we cut the complex plane along the interval $[x_0, x_k]$ of the real axis and define a function $f$ of one complex variable $z$ in terms of a multivariate hypergeometric $R$-function (the Dirichlet average of the power function) [3, Sect. 5.9]

$$f(z) = R_{-1}(1, 1, ..., 1; z - x_0, z - x_1, ..., z - x_k). \tag{1.2}$$

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The knots $x_0, \ldots, x_k$ are logarithmic branch points of $f$ (see (4.4)), but $f$ is single-valued and analytic in the cut plane and approaches $1/z$ as $|z| \to \infty$. If $x$ is on the cut, the value of $f$ on the lower edge of the cut at $x$ is the limit as $\varepsilon \downarrow 0$ of $f(x - i\varepsilon)$, the value of $f$ on the upper edge of the cut is the limit of $f(x + i\varepsilon)$, and the difference of these two limits is the jump of $f$ across the cut. By two methods in Sections 2 and 3 we shall prove

THEOREM 1. *For every real x,*

$$(1/2)[M(x+0)+M(x-0)] = (1/2\pi i) \lim_{\varepsilon \downarrow 0} [f(x-i\varepsilon) - f(x+i\varepsilon)]. \tag{1.3}$$

The equality is obvious at a real point $x$ not on the cut because $f$ and $M$ are continuous there and $M(x)=0$. For every real $x$, $M(x)$ is $1/2\pi i$ times the jump of $f$ across the real axis at $x$ if $M$ is continuous at that point. In Section 4 it is observed that some properties of $B$-splines follow from properties of the $R$-function, but a more direct method is to represent a $B$-spline as a Dirichlet average of a Dirac delta function. In Section 5 the theory of Dirichlet averages [3, Chap. 5] is modified to allow for vector variables, and the results are then used in Section 6 to obtain some new and some old properties of multivariate $B$-splines with possibly repeated knots.

## 2. $B$-SPLINES AS JUMP DISCONTINUITIES

Let $E_k$ be the standard $k$-simplex defined by $u_i \geqslant 0$, $0 \leqslant i \leqslant k$, and $\sum_{i=0}^{k} u_i = 1$. Since the $k$-volume of the simplex is $1/k!$, we define $d\mu(u) = k!\, du_1 \cdots du_k$ so that

$$\int_{E_k} d\mu(u) = 1. \tag{2.1}$$

Comparison of the Peano and Hermite-Genocchi representations of a divided difference (see Curry and Schoenberg [5, (1.5), (2.1)]) leads to the relation

$$\int_{-\infty}^{\infty} g(t)\, M(t \mid x_0, \ldots, x_k)\, dt = \int_{E_k} g\left(\sum_{i=0}^{k} u_i x_i\right) d\mu(u), \tag{2.2}$$

where $g$ is continuous on $[x_0, x_k]$. The case $g \equiv 1$ is (1.1). The limits of integration on the left side could be replaced by $x_0$ and $x_k$ since $M = 0$ outside the interval $[x_0, x_k]$. Equation (2.2), when required to hold for all continuous functions $g$, is sometimes taken as a functional definition of $M$. Comparison with [3, (4.4-1)] shows that $\mu$ is a Dirichlet measure (or multivariate beta distribution) $\mu_b$ with $b=(1, 1, ..., 1)$. In the notation of Dirichlet averages [3, (5.2-1)], (2.2) can be written as

$$\int_{-\infty}^{\infty} g(t)\, M(t \mid x_0, ..., x_k)\, dt = G(1, ..., 1; x_0, ..., x_k). \tag{2.3}$$

We now choose $g(t)=(z-t)^{-1}$, where $z$ is a point of the complex plane cut along the interval $[x_0, x_k]$ of the real axis. Since

$$\sum_{i=0}^{k} u_i = 1 \qquad \text{and} \qquad z - \sum_{i=0}^{k} u_i x_i = \sum_{i=0}^{k} u_i (z - x_i),$$

the second integral in (2.2) becomes

$$\int_{E_k} \left[ \sum_{i=0}^{k} u_i (z-x_i) \right]^{-1} d\mu(u). \tag{2.4}$$

Comparison with [3, (5.9-1)] shows that this Dirichlet average is a hypergeometric $R$-function

$$\int_{-\infty}^{\infty} (z-t)^{-1}\, M(t \mid x_0, ..., x_k)\, dt$$
$$= R_{-1}(1, ..., 1; z-x_0, ..., z-x_k). \tag{2.5}$$

It is homogeneous of degree $-1$ in the variables $z-x_0, ..., z-x_k$, it is single-valued and analytic in the cut $z$-plane [3, Theorem 5.11-1], and it approaches $1/z$ as $|z| \to \infty$.

To prove Theorem 1.1 we denote the right side of (2.5) by $f(z)$, as in (1.2), and put $z=x \pm i\varepsilon$, where $x$ is real and $\varepsilon$ is positive. Then (2.5) implies

$$\frac{1}{2\pi i} [f(x-i\varepsilon) - f(x+i\varepsilon)]$$
$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} [(x-t-i\varepsilon)^{-1} - (x-t+i\varepsilon)^{-1}]\, M(t)\, dt$$
$$= \int_{-\infty}^{\infty} \lambda(t)\, M(t)\, dt, \tag{2.6}$$

where

$$\lambda(t) = \varepsilon/\pi[(x-t)^2+\varepsilon^2], \qquad \int_{-\infty}^{\infty} \lambda(t)\, dt = 1. \tag{2.7}$$

As $\varepsilon \downarrow 0$ we note that $\lambda(t) \to 0$ if $t \neq x$ while $\lambda(x) \to \infty$. Thus $\lambda$ becomes a Dirac delta function as $\varepsilon \downarrow 0$ [8, p. 35],

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} [f(x - i\varepsilon) - f(x + i\varepsilon)]$$

$$= \int_{-\infty}^{\infty} \delta(t - x)\, M(t)\, dt = M(x), \tag{2.8}$$

provided $M$ is continuous at $x$. If it is discontinuous, the last member of (2.8) is replaced by $[M(x+0) + M(x-0)]/2$ because $\delta$ is a limit of even functions. However, $M$ can be discontinuous only at $x_0$ or $x_k$ and only if $k$ of its $k+1$ knots coincide at that point.

Equation (2.8) is a Sokhotskyi–Plemelj formula for Cauchy integrals. For proofs in more general contexts, see [15, Sect. 17; 10, Sect. 14.1; 12, Theorem 3.1 and Sect. 5].

## 3. A Second Method

At a point where $M$ is continuous, an alternative route to (1.3) is the way it was discovered. Let $\gamma$ be a simple closed contour that encircles the cut in the positive direction, and let $g$ be analytic inside and on $\gamma$. The Dirichlet average of Cauchy's integral formula [3, (5.11-2)] shows that the right side of (2.3) is equal to

$$\frac{1}{2\pi i} \int_{\gamma} g(z)\, R_{-1}(1, ..., 1; z - x_0, ..., z - x_k)\, dz. \tag{3.1}$$

By Cauchy's integral theorem, $\gamma$ can be deformed into two line segments parallel to the cut at a distance $\varepsilon$ above and below it, connected by two semicircles of radius $\varepsilon$ centered on $x_0$ and $x_k$. Unless $k$ of the $k+1$ knots coincide at $x_0$ or $x_k$, one can show with the help of [3, (8.3-2)] that the integrals around the semicircles tend to 0 as $\varepsilon \downarrow 0$. The two line segments contribute

$$\frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_{x_0}^{x_k} [g(t - i\varepsilon)\, f(t - i\varepsilon) - g(t + i\varepsilon)\, f(t + i\varepsilon)]\, dt$$

$$= \frac{1}{2\pi i} \int_{x_0}^{x_k} g(t) \lim_{\varepsilon \downarrow 0} [f(t - i\varepsilon) - f(t + i\varepsilon)]\, dt, \tag{3.2}$$

where $f$ is defined by (1.2). Since $g$ is an arbitrary analytic function, comparison with the left side of (2.3) leads to (1.3).

## 4. Some Properties of Univariate B-Splines

We shall mention briefly how some known properties of univariate $B$-splines, as well as one property that is new, can be inferred from properties of the $R$-function by using Theorem 1. Details will be omitted because generalizations to multivariate $B$-splines will be presented later in this paper. A recurrence relation [1, p. 131, (5); 17, (4.22)] follows from Zill's recurrence relation [3, Exercise 5.9-6] for the $R$-function, but it is a special case of (6.11) below. A differential recurrence relation [1, p. 138, (9); 17, (4.23)] can be deduced from [3, Exercises 5.9-18, 5.9-6] but is a special case of (6.10) below. Since the $R$-function satisfies a system of Euler–Poisson differential equations [3, (5.4-2)], so does the $B$-spline (a new result), as shown more generally in Theorem 5.

The moments of a univariate $B$-spline [17, (4.39)] are much simpler than in the multivariate case (cf. (6.14)) and can be obtained by putting $g(t)=t^r$, $r=0, 1, 2, ...,$ in (2.2) and using [3, (5.7-1)]:

$$\int_{-\infty}^{\infty} t^r M(t \mid x_0, ..., x_k)\, dt = R_r(1, ..., 1; x_0, ..., x_k). \tag{4.1}$$

The hypergeometric polynomial on the right side is a normalized complete symmetric function of degree $r$ in the knots [3, (6.2-11)].

In (2.2) we choose $g=\lambda$, where $\lambda$ is defined by (2.7), and let $\varepsilon \downarrow 0$ to obtain

$$\begin{aligned} M(x \mid x_0, ..., x_k) &= \int_{E_k} \delta\left(x - \sum_{i=0}^{k} u_i x_i\right) d\mu(u) \\ &= \int_{E_k} \delta\left(\sum_{i=0}^{k} u_i (x - x_i)\right) d\mu(u), \end{aligned} \tag{4.2}$$

where $\delta$ is the Dirac delta function. The first equality says that $M(x)$ is proportional to the $(k-1)$-volume of the intersection of the hyperplane $\sum_{i=0}^{k} u_i x_i = x$ with the simplex $E_k$. A more detailed form of this statement is the geometrical interpretation due to Curry and Schoenberg [5, Theorem 2]. Since the convex combination $\sum_{i=0}^{k} u_i x_i$ is a point in the interval $[x_0, x_k]$ when $(u_1, ..., u_k)$ is a point in $E_k$, (4.2) makes it obvious that $M$ vanishes outside the closed interval and is positive in the open interval. Multiplying (4.2) by $g(x)$, integrating over all real $x$, and changing the order of integration on the right side, we recover (2.2).

Moreover, (4.2) shows that $M$ is the $k$th divided difference of a truncated power function [5, (1.4)], for the Hermite–Genocchi formula [5, (2.1)], [3, (5.5-6)] equates the last member of (4.2) with the $k$th divided difference at $x_0, ..., x_k$ of a Green's function $h(t)$ whose $k$th derivative is $k!\, \delta(x-t)$.

Up to an arbitrary polynomial of degree $k-1$, this implies $h(t)=0$ for $t<x$ and $h(t)=k(t-x)^{k-1}$ for $t>x$. Since the $k$th difference of the arbitrary polynomial is zero, $M$ is the $k$th divided difference of $h$. If $k=1$, $h$ is a Heaviside function with unit step at $x$. This agrees with the conclusion from (4.2) and [3, (5.5-2)(5.5-3)] that

$$\begin{aligned} M(x \mid x_0, x_1) &= (x_1-x_0)^{-1} \int_{x-x_1}^{x-x_0} \delta(t)\, dt \\ &= (x_1-x_0)^{-1} \qquad \text{if} \quad x_0<x<x_1, \\ M'(x \mid x_0, x_1) &= (x_1-x_0)^{-1} [\delta(x-x_0)-\delta(x-x_1)]. \end{aligned} \tag{4.3}$$

Similarly, the $R$-function in (1.2) is a $k$th divided difference at $x_0, ..., x_k$ of $q(t)=(-1)^k k(z-t)^{k-1} \log(z-t)$. The logarithm is to be taken real when its argument is positive. If $x_0, ..., x_k$ are all distinct, an explicit formula [3, Ex. 5.9-24] is

$$\begin{aligned} &R_{-1}(1, ..., 1; z-x_0, ..., z-x_k) \\ &\quad = k \sum_{i=0}^{k} (z-x_i)^{k-1} \log(z-x_i) \Big/ \prod_{j=0}^{k}{}' (x_j-x_i), \end{aligned} \tag{4.4}$$

where the prime signifies omission of the factor with $j=i$. If $k=1$ this reduces to

$$R_{-1}(1, 1; z-x_0, z-x_1) = (x_1-x_0)^{-1} \log[(z-x_0)/(z-x_1)], \tag{4.5}$$

and the case $(x_0, x_1)=(-1, 1)$ is the Legendre function $Q_0(z)$ of the second kind.

Two generalizations of (4.2) are useful. The first is replacement of the measure $\mu$ by a more general Dirichlet measure $\mu_b$; this allows one to simplify some recurrence relations and also is convenient when some of the knots coalesce. The second generalization is to replace the scalar argument of the Dirac delta function by a vector in preparation for multivariate $B$-splines. For this reason we shall discuss Dirichlet averages of a function of a vector variable before returning to $B$-splines in Section 6.

## 5. Dirichlet Averages with Vector Variables

The measure $\mu$ in (2.1) and (4.2) is the case $b=(1, ..., 1)$ of a more general Dirichlet measure [3, Sect. 4.4] defined on the simplex $E_k$,

$$\begin{aligned} d\mu_b(u) &= \Gamma\left(\sum_{i=0}^{k} b_i\right) \prod_{i=0}^{k} (u_i^{b_i-1}/\Gamma(b_i))\, du_1 \cdots du_k, \\ &\int_{E_k} d\mu_b(u) = 1, \end{aligned} \tag{5.1}$$

where $\Gamma$ is the gamma function and the components of $b=(b_0, ..., b_k)$ are positive or even complex with positive real parts. Let $f$ be a function of a vector variable $x \in \mathbb{R}^s$ with components $x_1, ..., x_s$, and assume $f$ is either a function with continuous second derivatives or a generalized function. The Dirichlet average of $f$ is

$$F(b, z) = F(b_0, ..., b_k; z_0, ..., z_k) = \int_{E_k} f\left(\sum_{i=0}^{k} u_i z_i\right) d\mu_b(u), \tag{5.2}$$

where each $z_i$ is a vector in $\mathbb{R}^s$ with components $z_{i1}, ..., z_{is}$. The function $F(b_0, ..., b_k; z_0, ..., z_k)$ is symmetric in the indices $0, ..., k$ [3, Theorem 5.2-3]. A rule [3, (5.2-3)] that will be useful later in connection with coalescent knots is that equal $z$-variables can be replaced by a single variable if the corresponding $b$-parameters are replaced by their sum, e.g.,

$$\begin{aligned} &F(b_0, b_1, b_2, ..., b_k; z_1, z_1, z_2, ..., z_k) \\ &\quad = F(b_0 + b_1, b_2, ..., b_k; z_1, z_2, ..., z_k). \end{aligned} \tag{5.3}$$

Also, the requirement that the $b$-parameters have positive real parts can be relaxed by analytic continuation, and a vanishing $b$-parameter can then be omitted along with its corresponding $z$-variable [3, (6.3-3)], e.g.,

$$F(0, b_1, ..., b_k; z_0, z_1, ..., z_k) = F(b_1, ..., b_k; z_1, ..., z_k). \tag{5.4}$$

Define $c = \sum_{i=0}^{k} b_i$ and $w_i = b_i/c$, whence $\sum_{i=0}^{k} w_i = 1$. Let $e_i$ denote a $(k+1)$-tuple with $i$th component unity and all other components zero, e.g., $b - e_0 = (b_0 - 1, b_1, ..., b_k)$. The relation [3, (5.6-4)] between associated functions holds unchanged when the $z_i$ are vectors:

$$F(b, z) = \sum_{i=0}^{k} w_i F(b + e_i, z). \tag{5.5}$$

This relation is an immediate consequence of [3, (5.6-8)], which leads also to

$$g(x) = x f(x) \Rightarrow G(b, z) = \sum_{i=0}^{k} w_i z_i F(b + e_i, z), \tag{5.6}$$

where $x$, $g$, $z_i$, and $G$ are vectors.

If $f_m = \partial f / \partial x_m$, $m = 1, ..., s$, we define

$$F_m(b, z) = \int_{E_k} f_m\left(\sum_{i=0}^{k} u_i z_i\right) d\mu_b(u), \tag{5.7}$$

and similarly $F_{mn}$ denotes the Dirichlet average of $f_{mn} = \partial^2 f/\partial x_m\, \partial x_n$. If $s=1$, $F_1$ is denoted in [3] by $F'$ and $F_{11}$ by $F''$. The differentiations performed in establishing differential properties in [3] are still permissible when smooth functions are replaced by generalized functions [8, Chap. III]. If $D_{im} = \partial/\partial z_{im}$ we see that

$$D_{im} F(b, z) = \int_{E_k} f_m\left(\sum_{i=0}^{k} u_i z_i\right) u_i\, d\mu_b(u) = w_i F_m(b + e_i, z), \tag{5.8}$$

where we have used [3, (5.6-8)]. If $i \neq j$ iteration of (5.8) gives

$$\begin{aligned} D_{im} D_{jn} F(b, z) &= D_{jm} D_{in} F(b, z) \\ &= \frac{b_i b_j}{c(c+1)} F_{mn}(b + e_i + e_j, z). \end{aligned} \tag{5.9}$$

From (5.8) and (5.5) it follows that

$$\begin{aligned} \sum_{i=0}^{k} D_{im} F(b, z) &= F_m(b, z), \\ \sum_{i=0}^{k} D_{im} F_n(b, z) &= F_{mn}(b, z), \end{aligned} \tag{5.10}$$

while (5.6) and (5.8) imply

$$h(x) = x_m f_n(x) \Rightarrow H(b, z) = \sum_{i=0}^{k} z_{im} D_{in} F(b, z). \tag{5.11}$$

Minor modifications in the procedure of [3, Sect. 5.4] lead to generalized Euler-Poisson equations:

THEOREM 2. *For $i, j = 0, 1, \ldots, k$ and $n = 1, \ldots, s$,*

$$\left[\sum_{m=1}^{s} (z_{im} - z_{jm})\, D_{im} D_{jn} + b_i D_{jn} - b_j D_{in}\right] F(b, z) = 0. \tag{5.12}$$

From these equations, by the same procedure used for $s=1$ in [3, Sect. 5.6] except that now $G_n = F$, we obtain

THEOREM 3. *For $i = 0, \ldots, k$ and $c = \sum_{i=0}^{k} b_i$,*

$$\begin{aligned} &\sum_{m=1}^{s} z_{im} F_m(b, z) + (c-1)\, F(b - e_i, z) \\ &\quad = \sum_{j=0}^{k} \sum_{m=1}^{s} z_{jm} D_{jm} F(b, z) + (c-1)\, F(b, z). \end{aligned} \tag{5.13}$$

In the first step of the proof it is shown that the left side of (5.13) is independent of $i$. Multiplying by $\mu_i$ or $\lambda_i$ and summing over $i$, we get two corollaries:

COROLLARY 1. *If* $\sum_{i=0}^{k} \mu_i = 0$, *then*

$$\sum_{i=0}^{k} \sum_{m=1}^{s} \mu_i z_{im} F_m(b, z) + (c-1) \sum_{i=0}^{k} \mu_i F(b - e_i, z) = 0. \tag{5.14}$$

COROLLARY 2. *If* $\sum_{i=0}^{k} \lambda_i = 1$ *and* $\sum_{i=0}^{k} \lambda_i z_i = 0$, *then*

$$(c-1) \sum_{i=0}^{k} \lambda_i F(b - e_i, z) = \sum_{j=0}^{k} \sum_{m=1}^{s} z_{jm} D_{jm} F(b, z) + (c-1) F(b, z). \tag{5.15}$$

We consider now the Dirichlet averages

$$\Delta(b, z) = \int_{E_k} \delta\left(\sum_{i=0}^{k} u_i z_i\right) d\mu_b(u), \tag{5.16}$$

$$\Delta_m(b, z) = \int_{E_k} \delta_m\left(\sum_{i=0}^{k} u_i z_i\right) d\mu_b(u), \tag{5.17}$$

where $\delta(x) = \delta(x_1) \cdots \delta(x_s)$ is the Dirac delta function of a vector argument and $\delta_m(x) = (\partial/\partial x_m)\, \delta(x) = \delta(x_1) \cdots \delta(x_{m-1})\, \delta'(x_m)\, \delta(x_{m+1}) \cdots \delta(x_s)$. Because of the vector identity $x\delta(x) \equiv 0$, (5.6) becomes

$$\sum_{i=0}^{k} w_i z_i \Delta(b + e_i, z) = 0. \tag{5.18}$$

Differentiation of the scalar identity $x_m \delta(x) \equiv 0$ with respect to $x_n$ yields $\delta(m, n)\, \delta(x) + x_m \delta_n(x) = 0$, where $\delta(m, n)$ is the Kronecker delta. Taking Dirichlet averages with the help of (5.11), we find

$$\delta(m, n)\, \Delta(b, z) + \sum_{i=0}^{k} z_{im} D_{in} \Delta(b, z) = 0. \tag{5.19}$$

The case $m = n$ allows simplification of Theorem 3 and Corollary 2 when $F = \Delta$:

THEOREM 4. *For* $i = 0, ..., k$ *and* $c = \sum_{i=0}^{k} b_i$,

$$\sum_{m=1}^{s} z_{im} \Delta_m(b, z) + (c-1)\, \Delta(b - e_i, z) = (c-1-s)\, \Delta(b, z). \tag{5.20}$$

COROLLARY 3. *If* $\sum_{i=0}^{k} \lambda_i = 1$ *and* $\sum_{i=0}^{k} \lambda_i z_i = 0$, *then*

$$(c-1) \sum_{i=0}^{k} \lambda_i \Delta(b-e_i, z) = (c-1-s)\, \Delta(b, z). \tag{5.21}$$

## 6. MULTIVARIATE $B$-SPLINES AS DIRICHLET AVERAGES

A multivariate $B$-spline is a function of the vector variable $x \in \mathbb{R}^s$ with components $x_1, ..., x_s$. To avoid confusion with the components of $x$, denote the knots of the spline by $\xi = (\xi_0, ..., \xi_k)$, where each knot $\xi_i$ is a vector with components $\xi_{i1}, ..., \xi_{is}$. Let $b = (b_0, ..., b_k)$, where $b_i$ is the multiplicity of the knot $\xi_i$; for another treatment of repeated knots see [7]. The $B$-spline is

$$\begin{aligned} M(x) = M(x \mid \xi, b) &= \Delta(b_0, ..., b_k; x-\xi_0, ..., x-\xi_k) \\ &= \int_{E_k} \delta\left(x - \sum_{i=0}^{k} u_i \xi_i\right) d\mu_b(u), \end{aligned} \tag{6.1}$$

where $\Delta$ is defined in (5.16). It follows from (5.10) that

$$\begin{aligned} \partial M/\partial x_m &= -\sum_{i=0}^{k} (\partial/\partial \xi_{im})\, M(x \mid \xi, b) \\ &= \Delta_m(b_0, ..., b_k; x-\xi_0, ..., x-\xi_k). \end{aligned} \tag{6.2}$$

Equations (6.1) and (5.2) imply

$$\begin{aligned} &\int_{\mathbb{R}^s} g(x)\, M(x \mid \xi, b)\, dx_1 \cdots dx_s \\ &\quad = \int_{E_k} g\left(\sum_{i=0}^{k} u_i \xi_i\right) d\mu_b(u) = G(b, \xi) \end{aligned} \tag{6.3}$$

for any continuous function $g$ and in particular, if $g \equiv 1$,

$$\int_{\mathbb{R}^s} M(x \mid \xi, b)\, dx_1 \cdots dx_s = 1. \tag{6.4}$$

Since $\sum_{i=0}^{k} u_i \xi_i$ is a convex combination, (6.1) shows that $M(x) = 0$ if $x$ is outside the closed convex hull of $\{\xi_0, ..., \xi_k\}$.

Conditions on the number and position of the knots would be required to insure finiteness or continuity of $M$. These matters need not be discussed here; the equations of this section (except (6.16) and (6.18)) are valid without such conditions if $M$ is allowed to be a generalized function. For example, if all the knots coalesce, then $M(x)=\delta(x-\xi_0)$ by (6.1).

Because of (6.1) we can put $F=\Delta$ and $z_i=x-\xi_i$ in the equations of Section 5 to obtain properties of *B*-splines. Theorem 2 yields the new result that the dependence of a *B*-spline on its knots is governed by a system of Euler–Poisson equations:

THEOREM 5. *For $i, j=0, \ldots, k$ and $n=1, \ldots, s$,*

$$\left[\sum_{m=1}^{s} (\xi_{im}-\xi_{jm})(\partial^2/\partial\xi_{im}\,\partial\xi_{jn}) + b_i\,\partial/\partial\xi_{jn} - b_j\,\partial/\partial\xi_{in}\right] M(x \mid \xi, b)=0. \tag{6.5}$$

Equations (6.1), (5.8), and (6.2) imply

$$(\partial/\partial\xi_{im})\,M(x \mid \xi, b)= -w_i(\partial/\partial x_m)\,M(x \mid \xi, b+e_i), \tag{6.6}$$

where we recall that $w_i=b_i/c$ and $\sum_{i=0}^{k} w_i=1$. From (5.5) and (5.18) we find two previously known relations [13, p. 227; 7, (4.14)]:

$$\begin{aligned} M(x \mid \xi, b) &= \sum_{i=0}^{k} w_i M(x \mid \xi, b+e_i), \\ xM(x \mid \xi, b) &= \sum_{i=0}^{k} w_i \xi_i M(x \mid \xi, b+e_i). \end{aligned} \tag{6.7}$$

A new result comes from (5.19) with the help of (6.2):

THEOREM 6. *For $m, n=1, \ldots, s$,*

$$\left[\delta(m, n)+x_m\,\partial/\partial x_n+\sum_{i=0}^{k} \xi_{im}\,\partial/\partial\xi_{in}\right] M(x \mid \xi, b)=0. \tag{6.8}$$

Theorem 4 becomes

THEOREM 7. *For $i=0, \ldots, k$ and $c=\sum_{i=0}^{k} b_i$,*

$$\sum_{m=1}^{s} (x_m-\xi_{im})(\partial/\partial x_m)\,M(x \mid \xi, b)+(c-1)\,M(x \mid \xi, b-e_i) = (c-1-s)\,M(x \mid \xi, b). \tag{6.9}$$

This is equivalent to the last display equation on p. 82 of [11]; see also [9, (9); 7, (4.6)]. If $b_i=1$ note that the second term is independent of $\xi_i$

by (5.4). Corollaries 1 and 3 become relations due to Micchelli [14, pp. 493, 495; 11, (4.1)(4.2)]:

COROLLARY 4. *If* $\sum_{i=0}^{k} \mu_i = 0$ *the directional derivative of* $M$ *is*

$$\sum_{m=1}^{s} \sum_{i=0}^{k} \mu_i \xi_{im} (\partial/\partial x_m) M(x \mid \xi, b) = (c-1) \sum_{i=0}^{k} \mu_i M(x \mid \xi, b - e_i). \tag{6.10}$$

COROLLARY 5. *If* $\sum_{i=0}^{k} \lambda_i = 1$ *and* $\sum_{i=0}^{k} \lambda_i \xi_i = x$, *then*

$$(c-1-s)\, M(x \mid \xi, b) = (c-1) \sum_{i=0}^{k} \lambda_i M(x \mid \xi, b - e_i). \tag{6.11}$$

As observed in [6], multivariate $B$-splines are related to the double Dirichlet average [2, (2.10), (2.8)]

$$\mathscr{R}_t(b, Z, \beta) = \int_{E_k} R_t\left(\beta_1, ..., \beta_s; \sum_{i=0}^{k} u_i \xi_{i1}, ..., \sum_{i=0}^{k} u_i \xi_{is}\right) d\mu_b(u), \tag{6.12}$$

where $\beta = (\beta_1, ..., \beta_s)$ and $Z$ is a $(k+1) \times s$ matrix with elements $Z_{im} = \xi_{im}$. By (6.3) the moments of a multivariate $B$-spline are

$$\begin{aligned}
\int_{\mathbb{R}^s} \prod_{m=1}^{s} x_m^{r_m} M(x \mid \xi, b)\, dx_1 \cdots dx_s
&= \int_{E_k} \prod_{m=1}^{s} \left(\sum_{i=0}^{k} u_i \xi_{im}\right)^{r_m} d\mu_b(u) \\
&= \int_{E_k} R_{|r|}\left(-r_1, ..., -r_s; \sum_{i=0}^{k} u_i \xi_{i1}, ..., \sum_{i=0}^{k} u_i \xi_{is}\right) d\mu_b(u),
\end{aligned} \tag{6.13}$$

where $|r| = \sum_{m=1}^{s} r_m$ and we have used [3, (6.6-5)]. Comparison with (6.12) gives

THEOREM 8. *If* $\beta = (-r_1, ..., -r_s)$, $|r| = \sum_{m=1}^{s} r_m$, *and* $Z_{im} = \xi_{im}$, *then*

$$\int_{\mathbb{R}^s} \prod_{m=1}^{s} x_m^{r_m} M(x \mid \xi, b)\, dx_1 \cdots dx_s = \mathscr{R}_{|r|}(b, Z, \beta). \tag{6.14}$$

This is essentially the same as [6, (5.11)] because identical rows of $Z$ can be replaced by a single row if the corresponding $b$-parameters are replaced by their sum [2, p. 422].

The representation (1.3) of a univariate *B*-spline as a jump discontinuity has a somewhat complicated generalization. In the univariate case the Dirichlet average over $t$ of

$$\delta(x-t)=(1/2\pi i)\lim_{\varepsilon\downarrow 0}[(x-i\varepsilon-t)^{-1}-(x+i\varepsilon-t)^{-1}] \tag{6.15}$$

is

$$\begin{aligned}M(x\mid\xi,b)&=\Delta(b_0,...,b_k;x-\xi_0,...,x-\xi_k)\\&=(1/2\pi i)\lim_{\varepsilon\downarrow 0}[R_{-1}(b_0,...,b_k;x-i\varepsilon-x_0,...,x-i\varepsilon-x_k)\\&\qquad-R_{-1}(b_0,...,b_k;x+i\varepsilon-x_0,...,x+i\varepsilon-x_k)]\end{aligned} \tag{6.16}$$

if $M$ is continuous at $x$. The multivariate version of (6.15) is

$$\begin{aligned}\prod_{m=1}^{s}\delta(x_m-t_m)&\\&=(2\pi i)^{-s}\prod_{m=1}^{s}\lim_{\varepsilon_m\downarrow 0}[(x_m-i\varepsilon_m-t_m)^{-1}-(x_m+i\varepsilon_m-t_m)^{-1}]\\&=(2\pi i)^{-s}\lim_{\varepsilon_1\downarrow 0}\cdots\lim_{\varepsilon_s\downarrow 0}\sum_p(-1)^{|p|}\prod_{m=1}^{s}(x_m-(-1)^{p_m}i\varepsilon_m-t_m)^{-1},\end{aligned} \tag{6.17}$$

where the sum extends over all $p=(p_1,...,p_s)\in\{0,1\}^s$ and $|p|=\sum_{m=1}^{s}p_m$. The same steps that led to (6.14) now yield

THEOREM 9. *If $Z(p)$ is a $(k+1)\times s$ matrix with elements $Z_{jm}(p)=x_m-(-1)^{p_m}i\varepsilon_m-\xi_{jm}$ and if $\beta=(1,...,1)$ is an $s$-tuple, then*

$$M(x\mid\xi,b)=(2\pi i)^{-s}\lim_{\varepsilon_1\downarrow 0}\cdots\lim_{\varepsilon_s\downarrow 0}\sum_p(-1)^{|p|}\mathcal{R}_{-s}(b,Z(p),\beta), \tag{6.18}$$

*provided $M$ is continuous at $x$.*

The Fourier transform of a multivariate *B*-spline (cf. Micchelli [14, p. 496)] is obtained by putting $g(x)=e^{iy\cdot x}$ in (6.3), where $y\cdot x=\sum_{m=1}^{s}y_m x_m$, and

$$\begin{aligned}\int_{\mathbb{R}^s}e^{iy\cdot x}M(x\mid\xi,b)\,dx_1\cdots dx_s&\\&=\int_{E_k}\exp\left(i\sum_{j=0}^{k}u_j y\cdot\xi_j\right)d\mu_b(u)\\&=S(b_0,...,b_k;iy\cdot\xi_0,...,iy\cdot\xi_k),\end{aligned} \tag{6.19}$$

where $y \cdot \xi_j = \sum_{m=1}^{s} y_m \xi_{jm}$ and $S$ is a confluent hypergeometric function [3, Sect. 5.8]. In the univariate case with equally spaced knots $x_0, x_0+h, ..., x_0+kh$ of unit multiplicity, the Fourier transform reduces by [3, (5.8-3), Exercise 5.8-10] to

$$S(1, ..., 1; iyx_0, ..., iy(x_0+kh)) = \exp[iy(x_0+kh/2)] \left[\frac{\sin(yh/2)}{yh/2}\right]^k, \tag{6.20}$$

in agreement with [4, (1.1); 17, (4.65)] when $h=1$ and $x_0 = -k/2$.

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